

CLOSED RANGE OF $\bar{\partial}$ IN L^2 -SOBOLEV SPACES ON UNBOUNDED DOMAINS IN \mathbb{C}^n

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ABSTRACT. Let $\Omega \subset \mathbb{C}^n$ be a domain and $1 \leq q \leq n - 1$ fixed. Our purpose in this article is to establish a general sufficient condition for the closed range of the Cauchy-Riemann operator $\bar{\partial}$ in appropriately weighted L^2 -Sobolev spaces on $(0, q)$ -forms. The domains we consider may be neither bounded nor pseudoconvex, and our condition is a generalization of the classical $Z(q)$ condition that we call weak $Z(q)$. We provide examples that explain the necessity of working in weighted spaces both for closed range in L^2 and, even more critically, in L^2 -Sobolev spaces.

1. INTRODUCTION

This paper is a continuation of [HRb]. We suppose that $\Omega \subset \mathbb{C}^n$ is a smooth domain, and we require neither boundedness nor pseudoconvexity of Ω . Our objective is to find the weakest possible sufficient condition that ensures the Cauchy-Riemann operator $\bar{\partial}$ has closed range on $(0, q)$ -forms in L^2 -Sobolev spaces, for a *fixed* q , $1 \leq q \leq n - 1$. In [HRb], we proved closed range only in L^2 . When Ω is bounded and pseudoconvex, our result reproduces the classical cases (e.g., Kohn [Koh73]).

We continue to explore the weak $Z(q)$ hypothesis that we introduced in [HR15]. Weak $Z(q)$ (defined below) is a curvature condition on the Levi form that suffices to prove that the range of $\bar{\partial}$ is closed in $L^2_{0,q}$ or $L^2_{0,q+1}$ on bounded domains in Stein manifolds as well as unbounded domains with uniform C^3 regularity. The weak $Z(q)$ condition is a more general version than the authors' condition in [HR11], and is closely related to, but still more general than, related conditions in [Ho91], [ABZ06], and [Zam08] which have been investigated for closed range of $\bar{\partial}$ (or $\bar{\partial}_b$) in a variety of settings. Its name derives from the fact that it generalizes the classic $Z(q)$ condition (see [Hör65], [FK72], [AG62], or [CS01]).

Unbounded domains in \mathbb{C}^n may exhibit very different behavior than bounded ones. For example, Ω satisfies the classic $Z(q)$ condition when the Levi form has either at least $q + 1$ negative or at least $n - q$ positive eigenvalues at every boundary point. However, on any bounded domain, there must be at least one strictly (pseudo)convex boundary point, which forces (by continuity of the eigenvalues of the Levi form) a bounded $Z(q)$ domain in \mathbb{C}^n to have at least $n - q$ positive eigenvalues at every boundary point. Hence, a large class of interesting local examples (those with at least $q + 1$ negative eigenvalues) cannot be realized globally as bounded domains in \mathbb{C}^n (or indeed any Stein manifold). For an in depth look at the consequences of $Z(q)$ for unbounded domains, please see [HRa].

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In order to prove closed range of $\bar{\partial}$ in L^2 on any reasonable class of unbounded domains, it is necessary to work in weighted L^2 spaces. Unlike in the bounded case, these weighted L^2 spaces are not equivalent to the unweighted spaces. A simple counterexample demonstrates the necessity of using a weight function. Suppose that Ω contains balls of arbitrarily large radii. We want to see that the closed range estimate

$$(1.1) \quad \|u\|_{L^2(\Omega)} \leq C(\|\bar{\partial}u\|_{L^2(\Omega)} + \|\bar{\partial}^*u\|_{L^2(\Omega)})$$

cannot hold for any $C > 0$. Also $\bar{\partial}^*$ is the L^2 adjoint of $\bar{\partial}$ (see Section 2 for details on the notation). The Siegel upper space $\{(z, w) \in \mathbb{C}^{n+1} : \Im w > |z|^2\}$ satisfies the large ball condition and is the unbounded domain *par excellence* – its boundary is the Heisenberg group and it is also biholomorphic to the unit ball. By the large ball condition, there exists $z_R \in \Omega$ such that $B(z_R, R) \subset \Omega$ for every $R > 0$. Let $u_1 \in C_{0,(0,q)}^\infty(B(0, 1))$ be nontrivial, and define $u_R(z) = \frac{1}{R^n} u_1\left(\frac{z - z_R}{R}\right)$. Then $u_R \in C_{0,(0,q)}^\infty(B(z_R, R)) \subset C_{0,(0,q)}^\infty(\Omega)$. If (1.1) were to hold, then

$$\|u_1\|_{L^2(\Omega)} = \|u_R\|_{L^2(\Omega)} \leq C(\|\bar{\partial}u_R\|_{L^2(\Omega)} + \|\bar{\partial}^*u_R\|_{L^2(\Omega)}) = R^{-1}C(\|\bar{\partial}u_1\|_{L^2(\Omega)} + \|\bar{\partial}^*u_1\|_{L^2(\Omega)}).$$

Since this inequality must hold for every $R > 0$, we have a contradiction. Thus, closed range estimates in L^2 are impossible on many unbounded domains, so we must consider weighted L^2 spaces. In [HRb], we do briefly touch upon the L^2 -theory for $\bar{\partial}$ in unweighted L^2 spaces for domains that satisfy weak $Z(q)$. Gallagher and McNeal establish sufficient conditions for the closed range of $\bar{\partial}$ in L^2 unbounded, pseudoconvex domains [HM16].

Even if we wanted to concentrate on domains for which we can establish the unweighted L^2 theory for $\bar{\partial}$, there is no hope for any usable result in Sobolev spaces. The reason is that the Sobolev space theory is effectively useless on any interesting unbounded domain. For example, suppose that Ω contains infinitely many disjoint balls B_k of fixed radius r (as is the case in the model domain defined by $\rho(z) = \sum_{j=1}^n (\operatorname{Re} z_j)^2 - 1$ for which $\bar{\partial}$ has closed range in unweighted L^2 [HRb]). If we take any function $f \in C_0^\infty(B(0, r))$ and define $f_k(z) = f(z - c_k)$, where c_k is the center of B_k , then we have a sequence $\{f_k\}$ that is uniformly bounded in L^2 with no convergent subsequence. Hence, $H^1(\Omega)$ is not compact in $L^2(\Omega)$, and the Rellich Lemma fails, making any theory of Sobolev Spaces extremely problematic.

When working on weighted L^2 spaces for unbounded domains, adjoints of differential operators can introduce low order terms with unbounded coefficients. For example, if D is a differential operator and $e^{-\varphi}$ is our weight, we have

$$D_\varphi^* = e^\varphi D^* e^{-\varphi} = D^* + (\bar{D}\varphi).$$

Roughly speaking, our Sobolev spaces must be defined in such a way that multiplying by the unbounded function $\bar{D}\varphi$ is no worse than differentiating in D^* . This means that great care is required when defining Sobolev spaces. In [HR14], the authors developed the theory of weighted Sobolev spaces on unbounded domains building on ideas in [GH10] and [Gan]. Boundary smoothness also requires greater care, since derivatives of defining functions may still be unbounded even when the domain itself is smooth. In [HR13], the authors carefully examined defining functions for unbounded domains and concluded that from this perspective, the signed distance function works at least as well as any other defining function. To avoid undue technicalities, we will primarily use the weight $\varphi = t|z|^2$. Note that $t|z|^2$ will always satisfy (HII) – (HV) in [HR14].

With the tools of [HR15], [HR13], [HR14], and the L^2 theory established in [HRb], we are now able to prove closed range of the Cauchy-Riemann operator on appropriately defined Sobolev spaces for a large class of unbounded domains. We review our key definitions in Section 2. Section 3 recaps the proof of the basic estimate from [HRb]. We conclude the paper with the proof of the main theorem on Sobolev space in Section 4.

2. WEAKLY $Z(q)$ DOMAINS.

2.1. Notation. We follow the setup of [HRb]. Let $\Omega \subset \mathbb{C}^n$ be a domain with C^m boundary $\text{b}\Omega$.

Definition 2.1. We say that a defining function ρ for Ω is *uniformly C^m* if there exists an open neighborhood U of $\text{b}\Omega$ such that $\text{dist}(\text{b}\Omega, \text{b}U) > 0$, $\|\rho\|_{C^m(U)} < \infty$, and $\inf_U |\nabla \rho| > 0$.

There is no difference between uniform C^m and C^m on domains with compact boundary. On unbounded domains, however, we provided counterexamples, a large class of examples, and a complete characterization in terms of the signed distance function in [HR13].

We identify real $(1, 1)$ -forms with a hermitian matrix as follows:

$$c = \sum_{j,k=1}^n i c_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

For a function α , we denote $\alpha_k = \frac{\partial \alpha}{\partial z_k}$ and $\alpha_{\bar{j}} = \frac{\partial \alpha}{\partial \bar{z}_j}$.

Let $\rho : \mathbb{C}^n \rightarrow \mathbb{R}$ be a uniformly C^m -defining function for Ω . The L^2 -inner product on $L^2(\Omega, e^{-t|z|^2})$ is denoted by

$$(f, g)_t = \int_{\Omega} f \bar{g} e^{-t|z|^2} dV.$$

where dV is Lebesgue measure on \mathbb{C}^n . Let $d\sigma$ denote the induced surface area measure on $\text{b}\Omega$ and set $\|f\|_t^2 = \int_{\Omega} |f|^2 e^{-t|z|^2} dV$.

Let $\mathcal{I}_q = \{(i_1, \dots, i_q) \in \mathbb{N}^n : 1 \leq i_1 < \dots < i_q \leq n\}$. For $I \in \mathcal{I}_{q-1}$, $J \in \mathcal{I}_q$, and $1 \leq j \leq n$, let $\epsilon_j^I = (-1)^{|\sigma|}$ if $\{j\} \cup I = J$ as sets and $|\sigma|$ is the length of the permutation that takes $\{j\} \cup I$ to J . Set $\epsilon_j^I = 0$ otherwise. We use the standard notation that if $u = \sum_{J \in \mathcal{I}_q} u_J d\bar{z}_J$, then

$$u_{jI} = \sum_{J \in \mathcal{I}_q} \epsilon_j^I u_J.$$

Let $L_j^t = \frac{\partial}{\partial z_j} - t\bar{z}_j = e^{t|z|^2} \frac{\partial}{\partial z_j} e^{-t|z|^2}$ and let $\bar{\partial}_t^* : L_{0,q+1}^2(\Omega, e^{-t|z|^2}) \rightarrow L_{0,q}^2(\Omega, e^{-t|z|^2})$ be the L^2 -adjoint of $\bar{\partial} : L_{0,q}^2(\Omega, e^{-t|z|^2}) \rightarrow L_{0,q+1}^2(\Omega, e^{-t|z|^2})$. This means that if $f = \sum_{J \in \mathcal{I}_q} f_J d\bar{z}_J$ and $g = \sum_{K \in \mathcal{I}_{q+1}} g_K d\bar{z}_K \in \text{Dom}(\bar{\partial}_t^*)$, then

$$\bar{\partial} f = \sum_{\substack{J \in \mathcal{I}_q \\ K \in \mathcal{I}_{q+1}}} \sum_{k=1}^n \epsilon_K^{kJ} \frac{\partial f_J}{\partial \bar{z}_k} d\bar{z}_K \quad \text{and} \quad \bar{\partial}_t^* g = - \sum_{J \in \mathcal{I}_q} \sum_{j=1}^n L_j^t g_{jJ} d\bar{z}_J.$$

The induced CR-structure on $\text{b}\Omega$ at $z \in \text{b}\Omega$ is

$$T_z^{1,0}(\text{b}\Omega) = \{L \in T_z^{1,0}(\mathbb{C}) : \partial \rho(L) = 0\}.$$

Let $T^{1,0}(\text{b}\Omega)$ be the space of C^{m-1} sections of $T_z^{1,0}(\text{b}\Omega)$ and $T^{0,1}(\text{b}\Omega) = \overline{T^{1,0}(\text{b}\Omega)}$. We denote the exterior algebra generated by these spaces by $T^{p,q}(\text{b}\Omega)$.

Let ρ be a defining function so that $|d\rho| = 1$ on $\text{b}\Omega$. We define the *normalized Levi form* \mathcal{L} as the real element of $\Lambda^{1,1}(\text{b}\Omega)$ given by

$$\mathcal{L}(-iL \wedge \bar{L}) = i\partial\bar{\partial}\rho(-iL \wedge \bar{L})$$

for any $L \in T^{1,0}(\text{b}\Omega)$.

Definition 2.2. Given a set $M \subset \mathbb{C}^n$, a *tubular neighborhood* of M is an open set U_r of the form $U_r = \{p \in \mathbb{C}^n : \text{dist}(p, M) < r\}$ where $\text{dist}(\cdot, \cdot)$ is the Euclidean distance function. We call r the *radius* of U_r . If there exists $r > 0$ so that every point in U_r has a unique closest point in M , we say that M has *positive reach*.

2.2. Weak $Z(q)$ domains and closed range for $\bar{\partial}$. The following definition was introduced in [HR15], building on ideas in [HR11].

Definition 2.3. Let $\Omega \subset \mathbb{C}^n$ be a domain with a uniformly C^m defining function ρ , $m \geq 2$. We say $\text{b}\Omega$ (or Ω) satisfies $Z(q)$ *weakly* if there exists a hermitian matrix $\Upsilon = (\Upsilon^{\bar{k}j})$ of functions on $\text{b}\Omega$ that are uniformly bounded in C^{m-1} such that $\sum_{j=1}^n \Upsilon^{\bar{k}j} \rho_j = 0$ on $\text{b}\Omega$ and:

- (i) All eigenvalues of Υ lie in the interval $[0, 1]$.
- (ii) $\mu_1 + \dots + \mu_q - \sum_{j,k=1}^n \Upsilon^{\bar{k}j} \rho_{j\bar{k}} \geq 0$ where μ_1, \dots, μ_{n-1} are the eigenvalues of the Levi form \mathcal{L} in increasing order.
- (iii) $\inf_{z \in \text{b}\Omega} \{|q - \text{Tr}(\Upsilon)|\} > 0$.

Hörmander first used an identity, now called the basic identity, to prove a basic estimate for $\bar{\partial}$ on pseudoconvex domains [Hör65]. With our current hypotheses, we established the most general basic identity that we could formulate and it led to the definition of weak $Z(q)$ in [HR15, HRb]. Given suitable hypotheses, including $f \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$, $\text{b}\Omega$ is at least C^3 , and $i\partial\bar{\partial}\varphi = t i\partial\bar{\partial}|z|^2$ for some $t \in \mathbb{R}$, the basic identity we established in [HRb, Proposition 3.4] is

(2.1)

$$\begin{aligned} \|\bar{\partial}f\|_\varphi^2 + \|\bar{\partial}_\varphi^* f\|_\varphi^2 &= \sum_{J \in \mathcal{I}_q} \sum_{j,k=1}^n \left((I_{jk} - \Upsilon^{\bar{k}j}) \frac{\partial f_J}{\partial \bar{z}_k}, \frac{\partial f_J}{\partial \bar{z}_j} \right)_\varphi + \sum_{J \in \mathcal{I}_q} \sum_{j,k=1}^n \left(\Upsilon^{\bar{k}j} L_j^\varphi f_J, L_k^\varphi f_J \right)_\varphi \\ &+ \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n \int_{\text{b}\Omega} \langle \rho_{j\bar{k}} f_{jI}, f_{kI} \rangle e^{-\varphi} d\sigma - \sum_{J \in \mathcal{I}_q} \sum_{j,k=1}^n \int_{\text{b}\Omega} \langle \Upsilon^{\bar{k}j} \rho_{j\bar{k}} f_J, f_J \rangle e^{-\varphi} d\sigma \\ &+ 2 \text{Re} \left\{ \sum_{J \in \mathcal{I}_q} \sum_{j,k,\ell=1}^n \left(\frac{\partial \Upsilon^{\bar{k}j}}{\partial \bar{z}_k} \Upsilon^{\bar{\ell}j} L_\ell^\varphi f_J, f_J \right)_\varphi - \sum_{J \in \mathcal{I}_q} \sum_{j,k,\ell=1}^n \left(\frac{\partial \Upsilon^{\bar{k}j}}{\partial z_j} (I_{k\ell} - \Upsilon^{\bar{\ell}k}) \frac{\partial f_J}{\partial \bar{z}_\ell}, f_J \right)_\varphi \right\} \\ &+ \sum_{J \in \mathcal{I}_q} t((q - \text{Tr}(\Upsilon))f_J, f_J)_\varphi + O(\|f\|_\varphi^2), \end{aligned}$$

where $O(\|f\|_\varphi^2) \leq C(\|\Upsilon\|_{C^1} + \|\Upsilon\|_{C^2}^2)\|f\|_\varphi^2$ and I is the identity matrix.

The matrix Υ is chosen so that the boundary integral terms in the second line of (2.1) are nonnegative (and hence can be discarded) while keeping $\inf_{z \in \Omega} |q - \text{Tr} \Upsilon| > 0$. We also extended Υ into the interior of Ω (Lemma 3.1 below). The fact that the eigenvalues of Υ are nonnegative and bounded by 1 means that the terms in the first line of (2.1) are nonnegative. Finally, Property (iii) means that $((q - \text{Tr}(\Upsilon))f, f)_\varphi \sim \|f\|_\varphi^2$, allowing us to prove the basic

estimate, Proposition 3.3 below. The constant t is chosen large enough so that the junk terms in the third line of (2.1) are controlled, as is the $O(\|f\|_\varphi^2)$ term from the fourth line of (2.1).

For our results on weighted Sobolev spaces, an additional hypothesis is needed. In [HR14], we introduced six hypotheses (HI) – (HVI) that were important for developing the elliptic theory with weighted Sobolev spaces on unbounded domains. The first hypothesis was equivalent to Definition 2.1, so (HI) will be satisfied whenever we have a uniformly C^m defining function, $m \geq 3$. Hypotheses (HII) – (HV) are trivial for the weight function $\varphi = t|z|^2$, so we will not need to address them directly in this paper. Thus, we need only concern ourselves with (HVI). In the notation of the present paper, we have:

Definition 2.4. Let $\Omega \subset \mathbb{R}^d$ be an unbounded domain. We say Ω is *asymptotically non-radial* if

$$\inf_{r>0} \sup_{|x|>r, x \in \text{b}\Omega} \frac{x \cdot \nabla \rho}{|x| |\nabla \rho|} < 1$$

for any C^1 defining function ρ for Ω .

In [HR14], this condition is needed in order to show that the restrictions of our weighted Sobolev spaces to $\text{b}\Omega$ will still satisfy Rellich’s Lemma. A key step in the proof relies on the hypothesis that tangential derivatives of our weight function grow uniformly without bound. For the special weight function $|x|^2$, this is equivalent to Definition 2.4.

Geometrically, we are requiring that the normal vector is bounded away from the radial direction for sufficiently large $|x|$. To see that this is not a restrictive condition on unbounded domains, observe that $|x|$ can only increase very slowly in the boundary when the normal vector is almost radial. More precisely, for $r_0 > 0$ and $0 < \theta_1 - \theta_0 < 2\pi$ consider the unbounded open set in polar coordinates $U = \{(r, \theta) : r > r_0 \text{ and } \theta_0 < \theta < \theta_1\}$ and a domain $\Omega \subset \mathbb{R}^2$ defined in polar coordinates on U by $\Omega \cap U = \{(r, \theta) : r_0 < r < e^{f(\theta)}, \theta_0 < \theta < \theta_1\}$ for some $f \in C^1(\theta_0, \theta_1)$. Since Ω is defined on U by $\rho(r, \theta) = r - e^{f(\theta)}$, we have $\frac{x \cdot \nabla \rho}{|x| |\nabla \rho|} = (1 + (f'(\theta))^2)^{-1/2}$. Hence, Ω is unbounded and asymptotically nonradial near θ_0 on U if and only if

$$\lim_{\theta \rightarrow \theta_0^+} f(\theta) = \infty \text{ and } \limsup_{\theta \rightarrow \theta_0^+} f'(\theta) < 0.$$

Any rational function, for example, would satisfy this property. Constructing a counterexample that would also define a uniformly C^2 domain would require great care. Although more complicated behavior is possible in higher dimensions, it appears that asymptotic non-radiality is a mild restriction to make on a domain.

Before we state our main result, we prove a percolation result that greatly expands the scope of our main theorem.

Proposition 2.5. *Let $\Omega \subset \mathbb{C}^n$ be a domain with connected boundary that admits a uniformly C^2 defining function and satisfies weak $Z(q)$ for some $1 \leq q \leq n - 1$. If $q - \text{Tr}(\Upsilon) > 0$, then Ω satisfies weak $Z(q')$ for $q \leq q' \leq n - 1$. If $q - \text{Tr}(\Upsilon) < 0$, then Ω satisfies weak $Z(q')$ for $1 \leq q' \leq q$.*

Proof. The proof of the proposition follows easily from the fact that we may leave Υ unchanged and [HR15, Lemma 2.8]. This lemma says that weak $Z(q)$ with $q - \text{Tr} \Upsilon > 0$ implies that the Levi form of Ω has at least $(n - q)$ nonnegative eigenvalues and weak $Z(q)$ with

$q - \text{Tr } \Upsilon < 0$ implies that the Levi form has at least $(q + 1)$ nonpositive eigenvalues. The proof becomes transparent by diagonalizing the Levi form at a point (as we do immediately prior to [HR15, Lemma 2.8]) and inspecting the inequalities from the definition of weak $Z(q)$ in these coordinates. \square

The type of estimates that the weighted operators will satisfy is the following: for t sufficiently large, the operator T_t , initially known to be bounded from $L^2_{0,q'}(\Omega, e^{-t|z|^2})$ to $L^2_{0,q''}(\Omega, e^{-t|z|^2})$ for some q', q'' , will be shown to be continuous from $H^s_{0,q'}(\Omega, e^{-t|z|^2}, X)$ to $H^s_{0,q''}(\Omega, e^{-t|z|^2}, X)$ and satisfy the estimate

$$(2.2) \quad \|T_t u\|_{t,s,\Omega}^2 \leq C_s \|u\|_{t,s,\Omega}^2 + C_{t,s} \|u\|_t^2$$

where C_s only depends on s , $C_{t,s}$ depends on both t and s , and neither constant depends on u .

Theorem 2.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain with connected boundary that is asymptotically nonradial, admits a uniformly C^m defining function, $m \geq 3$, has positive reach, and satisfies weak $Z(q)$ for some $1 \leq q \leq n - 1$. Let $0 \leq s \leq m - 2$. Then there exists a $T_s > 0$ so that if $q - \text{Tr}(\Upsilon) > 0$ and $t \geq T_s$ or $q - \text{Tr}(\Upsilon) < 0$ and $t \leq -T_s$, then*

- (i) *The operator $\bar{\partial} : H^s_{0,\tilde{q}}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,\tilde{q}+1}(\Omega, e^{-t|z|^2}, X)$ has closed range for $\tilde{q} = q - 1$ or q ;*
- (ii) *The operator $\bar{\partial}_t^* : H^s_{0,\tilde{q}+1}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,\tilde{q}}(\Omega, e^{-t|z|^2}, X)$ has closed range for $\tilde{q} = q - 1$ or q ;*
- (iii) *The weighted $\bar{\partial}$ -Neumann Laplacian defined by $\square_{q,t} = \bar{\partial}\bar{\partial}_t^* + \bar{\partial}_t^*\bar{\partial}$ has closed range on $H^s_{0,q}(\Omega, e^{-t|z|^2}, X)$;*
- (iv) $\ker(\square_{q,t}) = \{0\}$.
- (v) *The following operators are continuous and satisfy estimates of type (2.2):*
 - (a) *The weighted $\bar{\partial}$ -Neumann operator $N_{q,t} : H^s_{0,q}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,q}(\Omega, e^{-t|z|^2}, X)$;*
 - (b) $\bar{\partial}_t^* N_{q,t} : H^s_{0,q}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,q-1}(\Omega, e^{-t|z|^2}, X)$
 - (c) $N_{q,t} \bar{\partial}_t^* : H^s_{0,q+1}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,q}(\Omega, e^{-t|z|^2}, X)$
 - (d) $\bar{\partial} N_{q,t} : H^s_{0,q}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,q+1}(\Omega, e^{-t|z|^2}, X)$
 - (e) $N_{q,t} \bar{\partial} : H^s_{0,q-1}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,q}(\Omega, e^{-t|z|^2}, X)$
 - (f) $\bar{\partial}_t^* N_{q,t} \bar{\partial} : H^s_{0,q-1}(\Omega, e^{-t|z|^2}, X) \rightarrow H^s_{0,q-1}(\Omega, e^{-t|z|^2}, X)$
 - (g) $\bar{\partial} \bar{\partial}_t^* N_{q,t}$ and $\bar{\partial}_t^* \bar{\partial} N_{q,t}$ mapping $H^s_{0,q}(\Omega, e^{-t|z|^2}, X)$ to itself.
- (vi) *If $\tilde{q} = q$ or $q + 1$ and $\alpha \in H^s_{0,\tilde{q}}(\Omega, e^{-t|z|^2}, X)$ so that $\bar{\partial}\alpha = 0$, then there exists $u \in H^s_{0,\tilde{q}-1}(\Omega, e^{-t|z|^2}, X)$ so that*

$$\bar{\partial}u = \alpha.$$

- (vii) *If $m = \infty$, $\tilde{q} = q$ or $q + 1$, and $\alpha \in C^\infty_{0,\tilde{q}}(\bar{\Omega})$ satisfies $\bar{\partial}\alpha = 0$, then there exists $u \in C^\infty_{0,\tilde{q}-1}(\bar{\Omega})$ so that*

$$\bar{\partial}u = \alpha.$$

Remark 2.7. The $s = 0$ case of Theorem 2.6 for parts (i) - (viii) is the main result in [HRb]. Also, the operator $\bar{\partial}_t^* N_{q,t}$ is the canonical solution operator for the $\bar{\partial}$ equation, and $(\bar{\partial} N_{q,t})^*$ is the canonical solution operator for the $\bar{\partial}$ -equation if $N_{q+1,t}$ exists. The latter operator may exist as a consequence of Proposition 2.5 and Theorem 2.6. Similarly, the operator $\bar{\partial} N_{q,t}$ is

the canonical solution operator for $\bar{\partial}_t^*$ on $(0, q)$ -forms and $(\bar{\partial}_t^* N_{q,t})^* = N_{q,t} \bar{\partial}$ is the canonical solution operator for $\bar{\partial}_t^*$ on $(0, q-1)$ -forms if $N_{q-1,t}$ exists. The operator $N_{q-1,t}$ will exist if $q - \text{Tr } \Upsilon < 0$.

Remark 2.8. We wish to point out a slight errata in Lemma 2.3 from [HR13]. A C^2 domain with positive reach must have a uniformly C^2 defining function, but the converse is not necessarily true. Consequently, Theorem 2.4 in [HRb] needs to include this hypothesis, as it relies on the results from [HR13].

Example 2.9. In [HRb], we show that for any $1 \leq p \leq n-1$ the quadric defined by

$$\rho(z) = \sum_{j=1}^p |z_j|^2 - \sum_{j=p+1}^n |z_j|^2 + 1$$

is a $Z(q)$ domain for any $q \neq n-p-1$ with a uniformly C^∞ defining function. One can easily check that such domains are also asymptotically non-radial.

3. THE BASIC ESTIMATE

In this paper, we will use the weight $\varphi = t|z|^2$, though we could also consider more general weight functions. For example, given a generic C^2 weight φ , the final (non-error term) in 2.1 would be

$$\sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n (\varphi_{j\bar{k}} f_{jI}, f_{kI})_\varphi - \sum_{J \in \mathcal{I}_q} \sum_{j,k=1}^n (\varphi_{j\bar{k}} \Upsilon^{\bar{k}j} f_J, f_J)_\varphi$$

The price of the more general weight is that we would have to change (iii) in Definition 2.3 to

$$\inf_{z \in b\Omega} \lambda_1 + \cdots + \lambda_q - \sum_{j,k=1}^n \varphi_{j\bar{k}} \Upsilon^{\bar{k}j} > 0,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\varphi_{j\bar{k}}$ arranged in increasing order (see the definition of q -compatible functions in [HR11]). We wish, however, to avoid this technicality. The basic estimate is the content of Proposition 3.3, and it quickly follows from the basic identity 2.1 and the following two lemmas from [HRb]. The first details the extension of Υ into the interior of Ω , and the second is a density lemma.

Lemma 3.1. *Suppose Ω has a connected boundary, a uniformly C^m defining function for some $m \geq 2$, and satisfies weak $Z(q)$ for some $1 \leq q \leq n-1$. Let Υ be as in Definition 2.3. There exists a hermitian matrix $\tilde{\Upsilon}$ of functions on \mathbb{C}^n that are uniformly bounded in C^{m-1} satisfying*

- (i) *All eigenvalues of $\tilde{\Upsilon}$ lie in the interval $[0, 1]$.*
- (ii) *$\tilde{\Upsilon}|_{b\Omega} = \Upsilon$, so that $\mu_1 + \cdots + \mu_q - \sum_{j,k=1}^n \tilde{\Upsilon}^{\bar{k}j} \tilde{\delta}_{j\bar{k}} \geq 0$ on $b\Omega$ where μ_1, \dots, μ_{n-1} are the eigenvalues of the Levi form in increasing order.*
- (iii) *$\inf_{z \in \bar{\Omega}} \{ |q - \text{Tr}(\tilde{\Upsilon})| \} > 0$.*
- (iv) *There exists $\epsilon > 0$ so that on the neighborhood U_ϵ of $b\Omega$ we have*

$$(3.1) \quad \sum_{j=1}^n \tilde{\Upsilon}^{\bar{k}j} \tilde{\delta}_j = 0.$$

We do not distinguish between Υ and its extension $\tilde{\Upsilon}$.

Lemma 3.2. *Let $\Omega \subset \mathbb{C}^n$ be a C^m domain, $m \geq 2$, and let $f \in L^2_{0,q}(\Omega, e^{-\varphi}) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$ for some C^2 function φ . Then there exists a sequence of bounded C^m domains $\{\Omega_j\}$ and functions $f_j \in C^{m-1}(\Omega)$ such that $\Omega_j \cap B(0, j+2) = \Omega \cap B(0, j+2)$, $f_j \equiv 0$ on $\Omega \setminus B(0, j+2)$, $f_j|_{\Omega_j} \in \text{Dom}(\bar{\partial}^*)$, and*

$$\begin{aligned} \|\bar{\partial}f_j\|_{L^2(\Omega_j, e^{-\varphi})} + \|\bar{\partial}^*f_j\|_{L^2(\Omega_j, e^{-\varphi})} + \|f_j\|_{L^2(\Omega_j, e^{-\varphi})} \\ \rightarrow \|\bar{\partial}f\|_{L^2(\Omega, e^{-\varphi})} + \|\bar{\partial}^*f\|_{L^2(\Omega, e^{-\varphi})} + \|f\|_{L^2(\Omega, e^{-\varphi})} \end{aligned}$$

Proposition 3.3. *Let Ω have a connected boundary, a uniformly C^m defining function for some $m \geq 2$, and satisfy weak $Z(q)$ for some $1 \leq q \leq n-1$. Suppose φ satisfies $\partial\bar{\partial}\varphi = t\partial\bar{\partial}|z|^2$. Then for any constant $\epsilon > 0$, there exists $T > 0$ so that if*

- (1) *either $t \leq -T$ and $(q - \text{Tr } \Upsilon) < 0$ or $t \geq T$ and $(q - \text{Tr } \Upsilon) > 0$, and*
- (2) *$f \in L^2_{0,q}(\Omega, e^{-\varphi}) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$,*

then

$$\epsilon (\|\bar{\partial}f\|_{\varphi}^2 + \|\bar{\partial}^*f\|_{\varphi}^2) \geq \|f\|_{\varphi}^2.$$

4. SOLVABILITY OF $\bar{\partial}$ IN $H^s_{(0,q)}(\Omega, e^{-t|z|^2})$

4.1. Definition of the Sobolev spaces $H^s(\Omega, e^{-t|z|^2})$. Define the weighted differential operators

$$X_j^t = \frac{\partial}{\partial x_j} - 2tx_j = e^{t|z|^2} \frac{\partial}{\partial x_j} e^{-t|z|^2}, \quad 1 \leq j \leq 2n$$

and

$$\nabla_X^t = (X_1^t, \dots, X_{2n}^t).$$

Definition 4.1. For a nonnegative $k \in \mathbb{Z}$, define the weighted Sobolev space

$$H^k(\Omega, e^{-t|z|^2}, X^t) = \{f \in L^2(\Omega, e^{-t|z|^2}) : (X^t)^\alpha f \in L^2(\Omega, e^{-t|z|^2}) \text{ for } |\alpha| \leq k\}$$

where $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is an $2n$ -tuple of nonnegative integers and

$$(X^t)^\alpha = (X_1^t)^{\alpha_1} \dots (X_{2n}^t)^{\alpha_{2n}}.$$

$H^k(\Omega, e^{-t|z|^2}, X^t)$ has the norm

$$\|f\|_{t,k,\Omega}^2 = \sum_{|\alpha| \leq k} \|(X^t)^\alpha f\|_t^2.$$

We suppress writing Ω when the domain is clear. Also, let

$$\begin{aligned} & H_0^k(\Omega, e^{-t|z|^2}, X^t) \\ &= \left\{ g \in H^k(\Omega, e^{-t|z|^2}, X^t) : \text{there exists } \{\psi_\ell\} \subset C_c^\infty(\Omega) \text{ satisfying } \lim_{\ell \rightarrow \infty} \|g - \psi_\ell\|_{t,k} = 0 \right\}. \end{aligned}$$

In other words, $H_0^k(\Omega, e^{-t|z|^2}, X^t)$ is the closure of $C_c^\infty(\Omega)$ in the $H^k(\Omega, e^{-t|z|^2}, X^t)$ -norm.

For $s > 0$, we define $H^s(\Omega, e^{-t|z|^2}, X^t)$ by real interpolation. The Sobolev space theory was worked out by the authors in [HR14]. As a consequence of Proposition 3.5 in [HR14], we have the following lemma.

Lemma 4.2. *Assume that Ω is asymptotically non-radial and has a uniformly C^2 defining function. Then $H_{0,q}^1(\Omega, e^{-|z|^2}, X^t) \subset \text{Dom}(\bar{\partial})$.*

4.2. Elliptic regularization. Before turning to the proof of Theorem 2.6 for $s > 0$, we need to do some preliminary work.

For $\epsilon > 0$, set

$$\begin{aligned} Q_t(u, v) &= (\bar{\partial}u, \bar{\partial}v)_t + (\bar{\partial}_t^*u, \bar{\partial}_t^*v)_t \\ Q_{t,\epsilon}(u, v) &= (\bar{\partial}u, \bar{\partial}v)_t + (\bar{\partial}_t^*u, \bar{\partial}_t^*v)_t + \epsilon(\nabla_X^t u, \nabla_X^t v)_t \end{aligned}$$

We can prove the elliptic regularity for $\square_{t,\epsilon} = \bar{\partial}_t^* \bar{\partial} + \bar{\partial} \bar{\partial}_t^* + \epsilon(\nabla_X^t)^* \nabla_X^t$.

Proposition 4.3. *Let Ω satisfy the hypotheses of Theorem 2.6. For $0 \leq s \leq m-2$, there exists a continuous operator $N_{q,t}^\epsilon : H_{0,q}^s(\Omega, e^{-|z|^2}, X^t) \rightarrow H_{0,q}^{s+2}(\Omega, e^{-|z|^2}, X^t) \cap \text{Dom}(\square_{q,t}^\epsilon)$ so that*

$$(4.1) \quad \square_{q,t}^\epsilon N_{q,t}^\epsilon u = u, \quad u \in H_{0,q}^s(\Omega, e^{-|z|^2}, X^t)$$

and

$$(4.2) \quad N_{q,t}^\epsilon \square_{q,t}^\epsilon u = u, \quad u \in \text{Dom}(\square_{q,t}^\epsilon) \cap H_{0,q}^s(\Omega, e^{-|z|^2}, X^t).$$

Proof. Using the notation of [HR14], we set $\mathcal{X} = H_{0,q}^1(\Omega, e^{-|z|^2}, X^t) \cap \text{Dom}(\bar{\partial}_t^*)$. We may use Proposition 3.3 to see that for sufficiently large $|t|$, given any $u \in L_{0,q}^2(\Omega, e^{-|z|^2})$, the map $v \mapsto (u, v)_t$ is a continuous, conjugate linear functional on \mathcal{X} since

$$|(u, v)_t| \leq \|u\|_t \|v\|_t \leq \frac{1}{C} \|u\|_t (Q_{t,\epsilon}(v, v))^{1/2}$$

where C depends on t but not on ϵ . Thus, by the Riesz Representation Theorem, there exists a unique $N_{q,t}^\epsilon u \in \mathcal{X}$ so that

$$(u, v)_t = Q_{t,\epsilon}(N_{q,t}^\epsilon u, v).$$

Moreover, $N_{q,t}^\epsilon u \in \text{Dom}(\square_{q,t}^\epsilon)$ (this is standard – see [Str10]), as are the equalities (4.1) and (4.2).

We now show that $N_{q,t}^\epsilon : H_{0,q}^s(\Omega, e^{-|z|^2}, X^t) \rightarrow H_{0,q}^{s+2}(\Omega, e^{-|z|^2}, X^t)$. Since $\text{Dom}(\bar{\partial}_t^*)$ is a closed subspace of $L_{0,q}^2(\Omega, e^{-|z|^2})$ and $(H_0^1)_{0,q}(\Omega, e^{-|z|^2}, X^t) \subset \text{Dom}(\bar{\partial}_t^*)$, it follows that

$$(H_0^1)_{0,q}(\Omega, e^{-|z|^2}, X^t) \subset \mathcal{X} \subset H_{0,q}^1(\Omega, e^{-|z|^2}, X^t).$$

Moreover, $Q_{t,\epsilon}(\cdot, \cdot)$ is strictly coercive over \mathcal{X} , so it follows from [HR14, Theorem 3.13] that $N_{q,t}^\epsilon : H_{0,q}^s(\Omega, e^{-|z|^2}, X^t) \rightarrow H_{0,q}^{s+2}(\Omega, e^{-|z|^2}, X^t)$ since $N_{q,t}^\epsilon u \in \mathcal{X}$ and $u \in H_{0,q}^s(\Omega, e^{-|z|^2}, X^t)$. \square

We now introduce the concept of a tangential operator. We follow the notation of [HR14, §4].

Definition 4.4. We call a first order differential operator T^t *weighted tangential* if there exists a vector field T so that $T^t = T - tT|z|^2$ and $T\rho = 0$. In other words, the principal part of T^t is tangential, and $(T^t)_t^* u = -Tu + O(u)$. If α is a multiindex and $(T^t)^\alpha = T_{\alpha_1}^t \cdots T_{\alpha_{|\alpha|}}^t$ where each $T_{\alpha_j}^t$ is tangential, then we say that $(T^t)^\alpha$ is *weighted tangential of order $|\alpha|$* .

Remark 4.5. It will be important that applying a (weighted) tangential derivative preserves $\text{Dom}(\bar{\partial}_t^*)$. In order to see this, we fix an atlas of boundary charts and define the action of a tangential derivative to a form expressed in the boundary coordinates to act componentwise. This will locally preserve $\text{Dom}(\bar{\partial}_t^*)$, and we can patch these together to obtain a global operator preserving $\text{Dom}(\bar{\partial}_t^*)$. If we express our form in other coordinates, this will only introduce lower order terms with C^{m-2} coefficients, and we will see that this causes no difficulty. For more details, see [CS01, Section 5.2] or the discussion in 2.3 of [Str10]. When we differentiate with respect to tangential derivatives below, we are implicitly doing so in a way that preserves $\text{Dom}(\bar{\partial}_t^*)$.

For a tangential operator T^α , we will want to estimate $Q_{t,\epsilon}((T^\alpha)^\epsilon N_{q,t}^\epsilon, (T^\alpha)^\epsilon N_{q,t}^\epsilon)$. To do so, we will need to work with slightly smoother forms. To that end, we prove the following density lemma that is a slight modification of [HR15, Lemma 4.1].

Lemma 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain with a uniformly C^m defining function, $m \geq 3$, and let $u \in H_{0,q}^1(\Omega, e^{-t|z|^2}, X^t) \cap \text{Dom}(\bar{\partial}_t^*)$. For any integer k so that $2 \leq k \leq m-1$, there exists a sequence $u_\ell \in C_{0,q}^k(\bar{\Omega}) \cap H_{0,q}^k(\Omega, e^{-t|z|^2}, X^t) \cap \text{Dom}(\bar{\partial}_t^*)$ converging to u in the $H_{0,q}^1(\Omega, e^{-t|z|^2}, X^t)$ norm.*

Proof. Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cutoff function satisfying $\chi(x) \equiv 0$ on $(-\infty, 0]$ and $\chi(x) \equiv 1$ on $[1, \infty)$. For $r > 0$, let $u_r(z) = u(z)\chi\left(\frac{(r+1)^2 - |z|^2}{(r+1)^2 - r^2}\right)$. Observe that any fixed number of derivatives of $\chi\left(\frac{(r+1)^2 - |z|^2}{(r+1)^2 - r^2}\right)$ are uniformly bounded in r and supported in $B(0, r+1) \setminus \overline{B(0, r)}$. This means u_r is supported in $B(0, r+1)$ and u_r converges to u in $H_{0,q}^1(\Omega, e^{-t|z|^2}, X)$ as $r \rightarrow \infty$. Let Ω_r be a bounded C^m domain satisfying $\Omega_r \cap B(0, r+1) = \Omega \cap B(0, r+1)$. By a straight forward adaptation of Lemma 4.1 in [HR15], we can build a sequence $\{u_{\ell,r}\} \subset C_{0,q}^k(\bar{\Omega}_r) \cap \text{Dom}(\bar{\partial}_t^*)$ converging to u_r on Ω_r with respect to $W_{0,q}^1(\Omega_r)$. Multiplying again by our cutoff function gives us $u_{\ell,r}(z)\chi\left(\frac{(r+1)^2 - |z|^2}{(r+1)^2 - r^2}\right) \in C_{0,q}^k(\bar{\Omega}) \cap H_{0,q}^k(\Omega, e^{-t|z|^2}, X^t) \cap \text{Dom}(\bar{\partial}_t^*)$, and we can extract a convergent subsequence by taking r and ℓ sufficiently large. \square

For the next lemma, we need to use special boundary charts. Let T_1, \dots, T_{n-1} be an orthonormal basis of $(1,0)$ vector fields near $\text{b}\Omega$ so that $T_j \rho = 0$ on $\text{b}\Omega$. Let T_n be the vector field so that T_n is orthogonal to T_1, \dots, T_{n-1} , $D_\nu := \text{Re } T_n = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \nu}$ and $T_\nu := \text{Im } T_n$ is tangential near $\text{b}\Omega$ and orthogonal to T_1, \dots, T_{n-1} . Let $\bar{\omega}^1, \dots, \bar{\omega}^n$ be the dual basis. If $\text{b}\Omega$ has a uniformly C^m defining function, then $\bar{\omega}^j$ has coefficients (when expressed in the global coordinates $d\bar{z}^1, \dots, d\bar{z}^n$) that are uniformly C^{m-1} . Therefore, $\bar{\partial}\bar{\omega}^j$ has coefficients that are uniformly C^{m-2} .

In the special boundary chart, a $(0,q)$ -form u can be expressed as $u = \sum_{J \in \mathcal{I}_q} u_J \bar{\omega}^J$. Moreover, u has

$$\bar{\partial}u = \sum_{J \in \mathcal{I}_q} \sum_{k=1}^n \bar{T}_k u_J \bar{\omega}^k \wedge \bar{\omega}^J + O(u) \quad \text{and} \quad \vartheta^t u = - \sum_{I \in \mathcal{I}_{q-1}} \sum_{j=1}^n (T_j^t) u_{jI} \bar{\omega}^I + O(u)$$

where $u_{jI} = \sum_{J \in \mathcal{I}_q} \epsilon_j^I u_J$. Note that in the formula for $\vartheta^t u$, the error term is $O(u)$, not $O_t(u)$. This is due to the fact that only the first order component of a weighted derivative

satisfies the Leibniz formula, so, for example,

$$T^t(fg) = gT^t f + fTg.$$

The normal derivative D_ν is defined for z satisfying $\text{dist}(z, \text{b}\Omega) < \text{Reach}(\text{b}\Omega)$ and has coefficients that are uniformly C^{m-1} .

Lemma 4.7. *Let $\Omega \subset \mathbb{C}^n$ be a domain with a uniformly C^m defining function, $m \geq 3$. Let $u = \sum_{J \in \mathcal{I}_q} u_J \bar{\omega}^J$ be a $(0, q)$ -form defined near $\text{b}\Omega$. Let $J \in \mathcal{I}_q$.*

- (1) *If $n \notin J$, then we can express $\frac{\partial u_J}{\partial \nu}$ as a linear combination of coefficients of $\bar{\partial}u$, tangential derivatives of u , and u . The coefficients of the elements of $\bar{\partial}u$ and tangential derivatives of u are uniformly C^{m-1} , and the coefficient of u is uniformly C^{m-2} .*
- (2) *If $n \in J$, then the weighted normal derivative $D_\nu^t u_J$ can be expressed as a linear combination of coefficients of $\vartheta^t u$, weighted tangential derivatives of u , and u . The coefficients of the elements of ϑu and the weighted tangential derivatives of u are uniformly C^{m-1} , and the coefficient of u is uniformly C^{m-2} .*

Proof. Investigating $\bar{\partial}u$, observe that $k \notin J$ if and only if $\bar{\omega}^k \wedge \bar{\omega}^J \neq 0$. Consequently, if $n \notin J$, then the $\bar{\omega}^n \wedge \bar{\omega}^J$ component of $\bar{\partial}u$ is

$$\left(\bar{\partial}u\right)_{\bar{\omega}^n \wedge \bar{\omega}^J} = \bar{T}_n u_J + \sum_{\substack{J' \in \mathcal{I}_q \\ J' \neq J}} \sum_{k=1}^n \epsilon_{nJ}^{kJ'} \bar{T}_k u_{J'} + O(u).$$

Since $J' \neq J$, it follows that $k \neq n$ so that \bar{T}_k is a tangential vector field. Also, $\bar{T}_n u_J = D_\nu u_J - iT_\nu u_J$. Note then that if $n \notin J$, we have shown

$$D_\nu u_J = \left(\bar{\partial}u\right)_{\bar{\omega}^n \wedge \bar{\omega}^J} - \sum_{\substack{J' \in \mathcal{I}_q \\ J' \neq J}} \sum_{k=1}^{n-1} \epsilon_{nJ}^{kJ'} \bar{T}_k u_{J'} + iT_\nu u_J + O(u).$$

On the other hand when $n \in J$, we use $\vartheta^t u_J$ to control $D_\nu^t u_J$. Specifically, if $n \in J$ and $I = J \setminus \{n\}$, then $\epsilon_J^{nI} = (-1)^{q-1}$ and the $\bar{\omega}^I$ component of $\vartheta^t u$ is

$$\left(\vartheta^t u\right)_{\bar{\omega}^I} = -(-1)^{q-1} T_n^t u_J - \sum_{\substack{J' \in \mathcal{I}_q \\ J' \neq J}} \sum_{j=1}^{n-1} \epsilon_{J'}^{jI} T_j^t u_{J'} + O(u)$$

Each of the nonzero weighted derivatives $\epsilon_{J'}^{jI} T_j^t$ are weighted tangential. This means

$$(-1)^{q-1} D_\nu^t u_J = -\left(\vartheta^t u\right)_{\bar{\omega}^I} - \sum_{\substack{J' \in \mathcal{I}_q \\ J' \neq J}} \sum_{j=1}^{n-1} \epsilon_{J'}^{jI} T_j^t u_{J'} - i(-1)^{q-1} T_\nu^t u_J + O(u)$$

and the proof is complete. \square

We would like to remove the dichotomy in Lemma 4.7, namely, that some components are bounded with weighted tangential derivatives and ϑ^t and others by unweighted tangential derivatives and $\bar{\partial}$. However, we first record some technical lemmas about commutators of the various derivatives that appear.

Lemma 4.8. *Let $T^\alpha = T_{\alpha_1} \cdots T_{\alpha_\ell}$ be a tangential derivative of order $1 \leq \ell \leq m-1$ with coefficients that are uniformly C^{ℓ_1} , $\ell \leq \ell_1$. If X is a first order differential operator with coefficients that are uniformly $C^{\ell_2}(\Omega)$, $\ell \leq \ell_2$, then with $\ell_3 = \min\{\ell_1, \ell_2\}$ for every $\beta \subsetneq \alpha$, there exist first order operators X_β with coefficients that are uniformly $C^{\ell_3-(\ell-|\beta|)}$ such that*

$$(4.3) \quad [T^\alpha, X] = \sum_{\beta \subsetneq \alpha} T^\beta X_\beta.$$

Proof. The proof will follow from the computation that if $T = \sum_{j=1}^{2n} a_j \frac{\partial}{\partial x_j}$ and $X = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$, then

$$(4.4) \quad [T, X] = \sum_{j,k=1}^{2n} \left(a_j \frac{\partial b_k}{\partial x_j} - b_j \frac{\partial a_k}{\partial x_j} \right) \frac{\partial}{\partial x_k}.$$

Consequently, $[T, X]$ has coefficients that are uniform in C^{ℓ_3-1} .

Let $T^\alpha = T_{\alpha_1} \cdots T_{\alpha_\ell}$. Then $[T^\alpha, X] = T^\alpha X - X T^\alpha$ and expanding the commutator in more detail, we observe

$$\begin{aligned} T^\alpha X - X T^\alpha &= T^\alpha X - X T_{\alpha_1} \cdots T_{\alpha_\ell} = T^\alpha X - T_{\alpha_1} X T_{\alpha_2} \cdots T_{\alpha_\ell} + [X, T_{\alpha_1}] T_{\alpha_2} \cdots T_{\alpha_\ell} \\ &= T^\alpha X - (T_{\alpha_1} T_{\alpha_2} X + T_{\alpha_1} [X, T_{\alpha_2}] + T_{\alpha_2} [X, T_{\alpha_1}] + [[X, T_{\alpha_1}], T_{\alpha_2}]) T_{\alpha_3} \cdots T_{\alpha_\ell} \\ (4.5) \quad &= \sum_{\beta \subsetneq \alpha} T_{\beta_1} T_{\beta_2} \cdots T_{\beta_{|\beta|}} X_\beta \end{aligned}$$

where

$$(4.6) \quad X_\beta = [[[\cdots [[X, T_{(\alpha \setminus \beta)_1}], T_{(\alpha \setminus \beta)_2}], \cdots], T_{(\alpha \setminus \beta)_{\ell-k-1}}], T_{(\alpha \setminus \beta)_{\ell-|\beta|}}]$$

is an iterated commutator of X with $\ell - |\beta|$ tangential derivatives from T^α not included in T^β . We know that a commutator of two vectors fields with coefficients that are uniformly C^k produces a vector field with coefficients that are uniformly C^{k-1} . Since the iterated commutator defining X_β involves commuting X with $\ell - |\beta|$ vector fields, X_β is a vector field with uniformly $C^{\ell_3-(\ell-|\beta|)}$ coefficients. \square

Observe that $[X_j^t, X_k^t] = \left(\left[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right] \right)_t^* = 0$, so

$$(4.7) \quad [a X_j^t, b X_k^t] = a \frac{\partial b}{\partial x_j} X_k^t - b \frac{\partial a}{\partial x_k} X_j^t.$$

Using (4.7) to replace (4.4), we can repeat the argument of Lemma 4.8 to prove the following, weighted derivative version.

Lemma 4.9. *Let $T^\alpha = T_{\alpha_1} \cdots T_{\alpha_\ell}$ be a tangential derivative of order $1 \leq \ell \leq m-1$ with coefficients that are uniformly C^{ℓ_1} , $\ell \leq \ell_1$. If X is a first order differential operator with coefficients that are uniformly $C^{\ell_2}(\Omega)$, $\ell \leq \ell_2$, then with $\ell_3 = \min\{\ell_1, \ell_2\}$ for every $\beta \subsetneq \alpha$, there exists a first order weighted derivative X_β^t with coefficients that are uniformly $C^{\ell_3-(\ell-|\beta|)}$ so that*

$$(4.8) \quad [(T^\alpha)^t, X^t] = \sum_{\beta \subsetneq \alpha} (T^\beta)^t X_\beta^t.$$

□

Finally, we investigate the situation when X is not weighted but the tangential operators are weighted. The relevant commutator is $\left[X_j^t, \frac{\partial}{\partial x_k}\right] = \left[\frac{\partial}{\partial x_j} - 2tx_j, \frac{\partial}{\partial x_k}\right] = 2t\delta_{jk}$, so

$$(4.9) \quad [aX_j^t, b\frac{\partial}{\partial x_k}] = a\frac{\partial b}{\partial x_j}\frac{\partial}{\partial x_k} - b\frac{\partial a}{\partial x_k}X_j^t + 2abt\delta_{jk}.$$

Lemma 4.10. *Let $T^\alpha = T_{\alpha_1} \cdots T_{\alpha_\ell}$ be a tangential derivative of order $1 \leq \ell \leq m-1$ with coefficients that are uniformly C^{ℓ_1} , $\ell \leq \ell_1$. If X is a first order differential operator with coefficients that are uniformly $C^{\ell_2}(\Omega)$, $\ell \leq \ell_2$, then with $\ell_3 = \min\{\ell_1, \ell_2\}$ for every $\beta \subsetneq \alpha$, there exist a first order weighted derivative X_β^t , a vector field X'_β , and a function c_β so that*

$$(4.10) \quad [(T^\alpha)^t, X] = \sum_{\beta \subsetneq \alpha} (T^\beta)^t (X_\beta^t + X'_\beta + tc_\beta).$$

Moreover, X_β^t and X'_β have coefficients that are uniformly $C^{\ell_3-(\ell-|\beta|)}$ and c_β is a uniformly $C^{\ell_3-(\ell-|\beta|)+1}$ function.

Proof. The computation leading to (4.5) and (4.6) was formal, so in this case, we have

$$[X, (T^\alpha)^t] = \sum_{\beta \subset \alpha} (T_{\beta_1})^t (T_{\beta_2})^t \cdots (T_{\beta_k})^t X''_\beta$$

where

$$X''_\beta = [[[\cdots [X, T_{(\alpha \setminus \beta)_1}^t], T_{(\alpha \setminus \beta)_2}^t], \cdots], T_{(\alpha \setminus \beta)_{\ell-k-1}}^t], T_{(\alpha \setminus \beta)_{\ell-|\beta|}}^t]$$

We need to understand the terms in X''_β , and the iterated commutator of length $\ell - |\beta|$. We know that there exist Y_1 , Y'_1 , and c so that

$$[X, T_{(\alpha \setminus \beta)_1}^t] = Y_1^t + Y'_1 + tc$$

where Y_1^t and Y'_1 have coefficients that are uniformly C^{ℓ_3-1} and c is uniformly C^{ℓ_3} . Iterated commutators are linear in each of the components, so we can treat each piece separately. The iterated commutator piece with Y_1^t is handled with a repeated use of (4.7) to produce a weighted derivative with coefficients that are uniformly $C^{(\ell_3-1)-(\ell-|\beta|-1)}$, that is, uniformly $C^{\ell_3-(\ell-|\beta|)}$. The piece with tc is also relatively straight forward to handle. Since $[c, Y^t] = -(Yc) \cdot$, we see that the commutator of a function that is uniformly $C^{\ell'}$ with a weighted vector with uniformly $C^{\ell''}$ coefficients produces a function that is uniformly $C^{\min\{\ell', \ell''\}-1}$, hence the iterated commutator of length $\ell - |\beta| - 1$ involving tc produces a function tc_1 where c_1 is uniformly $C^{\ell_3-(\ell-|\beta|-1)}$.

Thus, it remains to handle

$$[[[\cdots [Y'_1, T_{(\alpha \setminus \beta)_2}^t], \cdots], T_{(\alpha \setminus \beta)_{\ell-k-1}}^t], T_{(\alpha \setminus \beta)_{\ell-|\beta|}}^t]],$$

an iterated commutator of length $\ell - |\beta| - 1$ of a vector field with uniformly C^{ℓ_3-1} coefficients with weighted tangential vector fields with uniformly C^{ℓ_2} coefficients. Repeating the argument we just completed (i.e., induction on the length of the iterated commutator) shows that we can write this commutator in the form

$$Y_2^t + Y'_2 + tc_2$$

where Y_2^t is a weighted derivative with coefficients that are uniformly $C^{\ell_3-1-(\ell-|\beta|-1)}$, i.e., uniformly $C^{\ell_3-(\ell-|\beta|)}$. Y_2' is a vector field, also with coefficients that are uniformly $C^{\ell_3-(\ell-|\beta|)}$ and c is a uniformly $C^{\ell_3-(\ell-|\beta|)+1}$ function. \square

A corollary of the previous three lemmas is that if α is a multindex of length k and $D^\alpha = D_{\alpha_1} \cdots D_{\alpha_k}$ where each D_{α_j} is either a vector field or weighted derivative that is tangential, respectively weighted tangential, near $\partial\Omega$ with uniformly C^ℓ coefficients, then for a vector field or weighted derivative X with coefficients that are uniformly C^{ℓ_1} , $k \leq \min\{\ell, \ell_1\}$,

$$[X, D^\alpha] = \sum_{\beta \subseteq \alpha} D^\beta (X_\beta^t + X_\beta' + tc_\beta)$$

where X_β^t and X_β' have uniformly $C^{\min\{\ell, \ell_1\}-(k-|\beta|)}$ coefficients and c_β has uniformly $C^{\min\{\ell, \ell_1\}-(k-|\beta|)+1}$ coefficients.

To show that $N_{q,t}^\epsilon$ is bounded in $H_{0,q}^k(\Omega, e^{-t|z|^2}, X^t)$ with a constant independent of $\epsilon > 0$, we need to bound

$$\|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 = \sum_{|\alpha| \leq k} \|(X^t)^\alpha N_{q,t}^\epsilon f\|_t^2.$$

The approach is to show that normal derivatives are controlled by $\bar{\partial}$, ϑ , and tangential derivatives and therefore we only need to bound tangential derivatives to control the full Sobolev norm. This is accomplished in the next proposition.

In a neighborhood of each boundary point, we can define ∇_T to be the vector with components $(\operatorname{Re} T_1, \operatorname{Im} T_1, \dots, \operatorname{Re} T_{n-1}, \operatorname{Im} T_{n-1}, T_\nu)$. By a partition of unity, we can extend ∇_T to a uniform neighborhood of the boundary. If we let $\nabla_T = \nabla_X$ on an interior set that is uniformly bounded away from the boundary, then we have a global gradient which differs from $\nabla_X = (X_1, \dots, X_{2n})$ in that ∇_T contains only derivatives in the tangential directions on a uniform neighborhood of the boundary. The following proposition extends Lemma 4.7 to higher order derivatives by showing that any k derivatives of v can be estimated in terms of $k-1$ derivatives of $\bar{\partial}v$, $k-1$ derivatives of $\vartheta^t v$, k weighted tangential derivatives of v , and lower order derivatives.

Proposition 4.11. *Let $v \in H_{0,q}^k(\Omega, e^{-t|z|^2}, X^t)$. Then there exist constants $C_k, C_{t,k} > 0$ so that*

$$(4.11) \quad \|v\|_{t,k,\Omega}^2 \leq C_k \left(\|\bar{\partial}v\|_{t,k-1,\Omega}^2 + \|\vartheta^t v\|_{t,k-1,\Omega}^2 + \|(\nabla_T^t)^k v\|_t^2 \right) + C_{t,k} \|v\|_{t,k-1,\Omega}^2.$$

Proof. We first assume that in some uniform neighborhood of the boundary, our k th order derivative takes the form $(D_\nu^t)^\ell (T^t)^\alpha$ where $|\alpha| + \ell = k$. Note that the ordering of the normal and tangential derivatives is irrelevant, since the commutators are of order at most $k-1$ and therefore bounded by (4.11). If $\ell = 0$, then the result is trivial, so we will proceed by induction on ℓ .

We next investigate the $\ell = 1, \alpha = 0$ case. Let v be a $(0, q)$ -form in $H_{0,q}^2(\Omega, e^{-t|z|^2}, X^t) \cap \operatorname{Dom}(\bar{\partial}_t^*)$. We collect some estimates before we carefully write down the estimate of the normal direction. Since T_j is tangential,

$$\|T_j v\|_t^2 = \|T_j^* v\|_t^2 + ([T_j^*, T_j]v, v)_t,$$

and it follows from (4.9) and the fact that the commutator of tangential derivatives is tangential that if T_j has uniformly C^k coefficients, then

$$\|T_j v\|_t^2 = \|T_j^t v\|_t^2 + (c_j \cdot \nabla_T v, v)_t + (c'_j \cdot \nabla_T^t v, v)_t + C_t \|v\|_t^2$$

and

$$\|T_j^t v\|_t^2 = \|T_j v\|_t^2 + (d_j \cdot \nabla_T v, v)_t + (d'_j \cdot \nabla_T^t v, v)_t + C_t \|v\|_t^2$$

where c_j, c'_j, d_j, d'_j are uniformly C^{k-1} if T_j has coefficients that are uniformly C_k . This means using a small constant/large constant argument and absorbing terms, that there exists $C_t > 0$ so that

$$(4.12) \quad \|\nabla_T v\|_t^2 \leq 2\|\nabla_T^t v\|_t^2 + C_t \|v\|_t^2$$

and

$$\|\nabla_T^t v\|_t^2 \leq 2\|\nabla_T v\|_t^2 + C_t \|v\|_t^2.$$

Write $v = v_1 + v_2$ where

$$v_1 = \sum_{\substack{J \in \mathcal{I}_q \\ n \in J}} v_J \bar{\omega}^J \quad \text{and} \quad v_2 = \sum_{\substack{J \in \mathcal{I}_q \\ n \notin J}} v_J \bar{\omega}^J,$$

in suitable local coordinates near each boundary point. In the interior, we let $v_1 = v$ and $v_2 = 0$. Turning to the normal derivatives themselves, we are now able to use Lemma 4.7 and establish

$$\begin{aligned} \|D_\nu v\|_t^2 + \|D_\nu^t v\|_t^2 &\leq 2(\|D_\nu v_1\|_t^2 + \|D_\nu^t v_1\|_t^2 + \|D_\nu v_2\|_t^2 + \|D_\nu^t v_2\|_t^2) \\ &\leq C(\|\vartheta^t v_1\|_t^2 + \|\nabla_T^t v_1\|_t^2 + \|\bar{\partial} v_2\|_t^2 + \|\nabla_T v_2\|_t^2 + \|v\|_t^2) \\ &\quad + \|D_\nu v_1\|_t^2 + \|D_\nu^t v_2\|_t^2. \end{aligned}$$

Note that $\bar{\partial}$ acts tangentially on v_1 and ϑ^t acts tangentially on v_2 , so we have

$$\begin{aligned} \|\bar{\partial} v_2\|_t^2 &\leq C(\|\bar{\partial} v\|_t^2 + \|\nabla_T v_1\|_t^2), \\ \|\vartheta^t v_1\|_t^2 &\leq C(\|\vartheta^t v\|_t^2 + \|\nabla_T^t v_2\|_t^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \|D_\nu v\|_t^2 + \|D_\nu^t v\|_t^2 &\leq C(\|\vartheta^t v\|_t^2 + \|\nabla_T^t v_1\|_t^2 + \|\bar{\partial} v\|_t^2 + \|\nabla_T v_2\|_t^2) + C_t \|v\|_t^2 \\ &\quad + C(\|\nabla_T v_1\|_t^2 + \|D_\nu v_1\|_t^2 + \|\nabla_T^t v_2\|_t^2 + \|D_\nu^t v_2\|_t^2). \end{aligned}$$

By [HR14, Proposition 3.5],

$$\begin{aligned} \|\nabla v_1\|_t^2 &\leq C\|\nabla_X^t v_1\|_t^2 + C_t \|v_1\|_t^2, \\ \|\nabla_X^t v_2\|_t^2 &\leq C\|\nabla v_2\|_t^2 + C_t \|v_2\|_t^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|D_\nu v\|_t^2 + \|D_\nu^t v\|_t^2 &\leq C(\|\vartheta^t v\|_t^2 + \|\nabla_T^t v_1\|_t^2 + \|\bar{\partial} v\|_t^2 + \|\nabla_T v_2\|_t^2) + C_t \|v\|_t^2 \\ &\quad + C(\|\nabla_T^t v_1\|_t^2 + \|D_\nu^t v_1\|_t^2 + \|\nabla_T v_2\|_t^2 + \|D_\nu v_2\|_t^2). \end{aligned}$$

Using Lemma 4.7 again together with (4.12), we obtain

$$\|D_\nu v\|_t^2 + \|D_\nu^t v\|_t^2 \leq C\left(\|\bar{\partial} v\|_t^2 + \|\vartheta^t v\|_t^2 + \|\nabla_T^t v\|_t^2\right) + C_t \|v\|_t^2.$$

We have shown that there exist constants $C, C_t > 0$ so that for every $v \in H_{0,q}^2(\Omega, e^{-t|z|^2}, X)$

$$(4.13) \quad \|D_\nu v\|_t^2 + \|D_\nu^t v\|_t^2 \leq C \left(\|\bar{\partial} v\|_t^2 + \|\vartheta^t v\|_t^2 + \|\nabla_T^t v\|_t^2 \right) + C_t \|v\|_t^2.$$

This proves the $k = 1$ case. For $k > 1$, (4.13) implies

$$\begin{aligned} & \|D_\nu(D_\nu^t)^{\ell-1}(T^t)^\alpha h\|_t^2 + \|(D_\nu^t)^\ell(T^t)^\alpha h\|_t^2 \\ & \leq C \left(\|\bar{\partial}(D_\nu^t)^{\ell-1}(T^t)^\alpha h\|_t^2 + \|\vartheta^t(D_\nu^t)^{\ell-1}(T^t)^\alpha h\|_t^2 + \|\nabla_T^t(D_\nu^t)^{\ell-1}(T^t)^\alpha h\|_t^2 \right) \\ & \quad + C_t \|(D_\nu^t)^{\ell-1}(T^t)^\alpha h\|_t^2 \\ & \leq C \left(\|(D_\nu^t)^{\ell-1}(T^t)^\alpha \bar{\partial} h\|_t^2 + \|(D_\nu^t)^{\ell-1}(T^t)^\alpha \vartheta^t h\|_t^2 + \|(D_\nu^t)^{\ell-1} \nabla_T^t (T^t)^\alpha h\|_t^2 \right) + C_{t,\epsilon} \|h\|_{t,k-1,\Omega}^2 \\ & \leq C \left(\|\bar{\partial} h\|_{t,k-1,\Omega}^2 + \|\vartheta^t h\|_{t,k-1,\Omega}^2 + \|(D_\nu^t)^{\ell-1} \nabla_T^t (T^t)^\alpha h\|_t^2 \right) + C_{t,\epsilon} \|h\|_{t,k-1,\Omega}^2. \end{aligned}$$

The third term can be estimated by our induction hypothesis on ℓ , and we are done. \square

Using Proposition 4.11, normal derivatives are controlled by tangential derivatives and we see that to control $\|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2$, it suffices to bound $\|(T^t)^\alpha N_{q,t}^\epsilon f\|_{t,k,\Omega}$ where $|\alpha| = k$, $(T^t)^\alpha = T_{\alpha_1}^t \cdots T_{\alpha_k}^t$, and each $T_{\alpha_j}^t$ is tangential near $\text{b}\Omega$. To do so, we must generalize [HR15, Lemma 4.2]. The issue is that the weighted terms are no longer benign in the sense that they cannot be treated separately as a lower order term from the first order part. As such, we investigate what the appropriate terms are that we need for control of the k th derivatives. Choose operators D_1, \dots, D_{4n} so that near $\text{b}\Omega$, $\{D_j : 1 \leq j \leq 2n\}$ are tangential and span the tangential directions and away from the boundary span \mathbb{C}^n . Let $\{D_j : 2n+1 \leq j \leq 4n\}$ be the weighted versions of first $2n$ vectors. For example, when $1 \leq j \leq 2n$ we may choose $D_j = \frac{\partial}{\partial x_j} - \frac{\partial \tilde{\delta}}{\partial x_j} \nabla \tilde{\delta} \cdot \nabla$ on a uniform neighborhood of the boundary and $D_j = \frac{\partial}{\partial x_j}$ away from this uniform neighborhood, with a suitable transition in between. Thus, either set can be used to obtain the Sobolev norms of a form in $\text{Dom}(\bar{\partial}_t^*)$, as long as we are willing to pay a price of multiplication by t in the lower order terms. We also assume that each D_j has uniformly C^{m-1} coefficients.

Let α be a multiindex of length k and $D^\alpha = D_{\alpha_1} \cdots D_{\alpha_k}$. Let $u_\alpha = D^\alpha N_{q,t}^\epsilon f$. By the discussion after Lemmas 4.8-4.10, we know that

$$\|[D^\alpha, \bar{\partial}_t^*] N_{q,t}^\epsilon f\|_t + \|[D^\alpha, \bar{\partial}] N_{q,t}^\epsilon f\|_t + \|[D^\alpha, \nabla_X] N_{q,t}^\epsilon f\|_t \leq C \|N_{q,t}^\epsilon f\|_{t,|\alpha|,\Omega} + C_t \|N_{q,t}^\epsilon f\|_{t,|\alpha|-1,\Omega}.$$

This means

$$\begin{aligned} Q_{t,\epsilon}(u_\alpha, u_\alpha) & \leq |(D^\alpha \bar{\partial} N_{q,t}^\epsilon f, \bar{\partial} u_\alpha)_t| + |(D^\alpha \bar{\partial}_t^* N_{q,t}^\epsilon f, \bar{\partial}_t^* u_\alpha)_t| + \epsilon |(D^\alpha \nabla_X N_{q,t}^\epsilon f, \nabla_X u_\alpha)_t| \\ & \quad + (C \|N_{q,t}^\epsilon f\|_{t,|\alpha|,\Omega} + C_t \|N_{q,t}^\epsilon f\|_{t,|\alpha|-1,\Omega}) \sqrt{Q_{t,\epsilon}(u_\alpha, u_\alpha)}. \end{aligned}$$

We would like to integrate by parts, but Proposition 4.3 gives us only that $\bar{\partial} u_\alpha, \bar{\partial}_t^* u_\alpha, \nabla_X u_\alpha$ are in $H_{(0,q)}^1(\Omega)$. Therefore, we use Lemma 4.6 to approximate u_α by $u_\alpha^\ell \in H_{(0,q)}^{|\alpha|+1}(\Omega) \cap$

$\text{Dom}(\bar{\partial}_t^*)$ and integrate by parts and commute again to obtain

$$\begin{aligned}
Q_{t,\epsilon}(u_\alpha, u_\alpha) &\leq \lim_{\ell \rightarrow \infty} \left[|(D^\alpha \bar{\partial} N_{q,t}^\epsilon f, \bar{\partial} u_\alpha^\ell)_t| + |(D^\alpha \bar{\partial}_t^* N_{q,t}^\epsilon f, \bar{\partial}_t^* u_\alpha^\ell)_t| + \epsilon |(D^\alpha \nabla_X N_{q,t}^\epsilon f, \nabla_X u_\alpha^\ell)_t| \right] \\
&\quad + (C \|N_{q,t}^\epsilon f\|_{t,|\alpha|,\Omega} + C_t \|N_{q,t}^\epsilon f\|_{t,|\alpha|-1,\Omega}) \sqrt{Q_{t,\epsilon}(u, u)} \\
&\leq \limsup_{\ell \rightarrow \infty} \left[|Q_{t,\epsilon}(N_{q,t}^\epsilon f, (D^\alpha)^* u_\alpha^\ell)_t| + |(\bar{\partial} N_{q,t}^\epsilon f, [(D^\alpha)^*, \bar{\partial}] u_\alpha^\ell)_t| \right. \\
&\quad \left. + |(\bar{\partial}_t^* N_{q,t}^\epsilon f, [(D^\alpha)^*, \bar{\partial}_t^*] u_\alpha^\ell)_t| + \epsilon |(\nabla_X N_{q,t}^\epsilon f, [(D^\alpha)^*, \nabla_X] u_\alpha^\ell)_t| \right] \\
&\quad + (C \|N_{q,t}^\epsilon f\|_{t,|\alpha|,\Omega} + C_t \|N_{q,t}^\epsilon f\|_{t,|\alpha|-1,\Omega}) \sqrt{Q_{t,\epsilon}(u_\alpha, u_\alpha)}.
\end{aligned}$$

By definition,

$$\limsup_{\ell \rightarrow \infty} Q_{t,\epsilon}(N_{q,t}^\epsilon f, (D^\alpha)^* u_\alpha^\ell)_t = \lim_{\ell \rightarrow \infty} (f, (D^\alpha)^* u_\alpha^\ell)_t = (D^\alpha f, u_\alpha)_t.$$

We have left to handle the commutator terms. They are estimated in the same fashion, and we show the estimate of $|(\bar{\partial} N_{q,t}^\epsilon f, [(D^\alpha)^*, \bar{\partial}] u_\alpha^\ell)_t|$. Recall that if T is a tangential operator with uniformly C^{m-1} coefficients, then $T^* = -T^t + c_T$ where c_T is a function that has uniformly C^{m-2} coefficients. If α' is defined so that $D_{\alpha_j}^* = D_{\alpha'_{k-j+1}} + c_{\alpha_j}$, then

$$(D^\alpha)^* = D_{\alpha_k}^* \cdots D_{\alpha_1}^* = (D_{\alpha'_1} + c_{\alpha_k}) \cdots (D_{\alpha'_k} + c_{\alpha_1}) = D^{\alpha'} + \sum_{\beta' \subsetneq \alpha'} D^{\beta'} c_{\beta'}.$$

Consequently, $[(D^\alpha)^*, \bar{\partial}]$ should be a form that we can control. In particular if $h = \sum_{J \in \mathcal{I}_q} h_J d\bar{z}^J$, then

$$[(D^\alpha)^*, \bar{\partial}]h = \sum_{J \in \mathcal{I}_q} \sum_{j=1}^n \left[(D^\alpha)^*, \frac{\partial}{\partial \bar{z}_j} \right] h_J d\bar{z}^j \wedge d\bar{z}^J,$$

and for appropriate first order operators $X_{\beta',j}$ and functions $c_{\alpha',j}$

$$\begin{aligned}
\left[(D^\alpha)^*, \frac{\partial}{\partial \bar{z}_j} \right] &= \left[D^{\alpha'} + \sum_{\beta' \subsetneq \alpha'} D^{\beta'} \tilde{c}_{\beta'}, \frac{\partial}{\partial \bar{z}_j} \right] \\
&= \sum_{|\beta| \leq k-1} D^\beta X_{\beta,j} + c_{\alpha',j}.
\end{aligned}$$

By an abuse of notation, we denote

$$[(D^\alpha)^*, \bar{\partial}]h = \sum_{J \in \mathcal{I}_q} \sum_{j=1}^n \sum_{|\beta| \leq k-1} (D^\beta X_{\beta,j} h_J + c_{\alpha',j} h_J) d\bar{z}^j \wedge d\bar{z}^J := \sum_{|\beta| \leq k-1} D^\beta X_\beta h + c_{\alpha'} h.$$

We now estimate

$$\begin{aligned}
\lim_{\ell \rightarrow \infty} |(\bar{\partial} N_{q,t}^\epsilon f, [(D^\alpha)^*, \bar{\partial}] u_\alpha^\ell)_t| &\leq \lim_{\ell \rightarrow \infty} \sum_{|\beta| \leq k-1} |(\bar{\partial} N_{q,t}^\epsilon f, (D^\beta X_\beta + c_{\alpha'}) u_\alpha^\ell)_t| \\
&= \sum_{|\beta| \leq k-1} |(\bar{\partial} N_{q,t}^\epsilon f, (D^\beta X_\beta + c_{\alpha'}) D^\alpha N_{q,t}^\epsilon f)_t| \\
&\leq \sum_{|\beta| \leq k-1} |((D^\beta)^* \bar{\partial} N_{q,t}^\epsilon f, X_\beta D^\alpha N_{q,t}^\epsilon f)_t| + \|N_{q,t}^\epsilon f\|_{t,1,\Omega} \|N_{q,t}^\epsilon f\|_{t,k,\Omega}
\end{aligned}$$

If $k = 1$, we are done. If $k > 1$, we cannot pass the X_β term to the other side, but we can commute X_β with D_{α_1} and integrate by parts to bring the D_{α_1} term across the inner product. Specifically,

$$\begin{aligned}
&|((D^\beta)^* \bar{\partial} N_{q,t}^\epsilon f, X_\beta D^\alpha N_{q,t}^\epsilon f)_t| \\
&= |(D_{\alpha_1}^* (D^\beta)^* \bar{\partial} N_{q,t}^\epsilon f, X_\beta D_{\alpha_2} \cdots D_{\alpha_k} N_{q,t}^\epsilon f)_t| + |((D^\beta)^* \bar{\partial} N_{q,t}^\epsilon f, [X_\beta, D_{\alpha_1}] D_{\alpha_2} \cdots D_{\alpha_k} N_{q,t}^\epsilon f)_t| \\
&\leq \left| \sum_{|\gamma| \leq |\beta|+1} (c_\gamma D^\gamma \bar{\partial} N_{q,t}^\epsilon f, X_\beta D_{\alpha_2} \cdots D_{\alpha_k} N_{q,t}^\epsilon f)_t \right| + C \|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 \\
&\leq C \left(\sum_{|\gamma| \leq |\beta|+1} \sqrt{Q_{t,\epsilon}(D^\gamma N_{q,t}^\epsilon f, D^\gamma N_{q,t}^\epsilon f)} \|N_{q,t}^\epsilon f\|_{t,k,\Omega} + \|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 \right).
\end{aligned}$$

Putting our estimates together, we have proven the following lemma.

Lemma 4.12. *Let $\Omega \subset \mathbb{C}^n$ satisfy the hypotheses of Theorem 2.6. There exist constants $C > 0$ independent of t and $C_t > 0$ depending on t such that for any $f \in C_{(0,q)}^\infty(\Omega)$ and operator D^α of order k that is a composition of operators that are tangential near $b\Omega$ and each have coefficients that are at least uniformly C^{m-1} , we have*

$$\begin{aligned}
(4.14) \quad &Q_{t,\epsilon}(D^\alpha N_{q,t}^\epsilon f, D^\alpha N_{q,t}^\epsilon f) \leq C |(D^\alpha f, D^\alpha N_{q,t}^\epsilon f)_t| \\
&+ C \left(\|N_{q,t}^\epsilon f\|_{t,|\alpha|,\Omega}^2 + \sum_{|\gamma| \leq k} \sqrt{Q_{t,\epsilon}(D^\gamma N_{q,t}^\epsilon f, D^\gamma N_{q,t}^\epsilon f)} \|N_{q,t}^\epsilon f\|_{t,k,\Omega} \right) + C_t \|N_{q,t}^\epsilon f\|_{t,|\alpha|-1,\Omega}^2.
\end{aligned}$$

Summing over all $|\alpha| \leq k$, we use Lemma 4.12 and obtain

$$\begin{aligned}
\sum_{|\alpha| \leq k} Q_{t,\epsilon}(D^\alpha N_{q,t}^\epsilon f, D^\alpha N_{q,t}^\epsilon f) &\leq C \sum_{|\alpha| \leq k} |(D^\alpha f, D^\alpha N_{q,t}^\epsilon f)_t| \\
&+ C \left(\|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 + \sum_{|\alpha| \leq k} \sqrt{Q_{t,\epsilon}(D^\alpha N_{q,t}^\epsilon f, D^\alpha N_{q,t}^\epsilon f)} \|N_{q,t}^\epsilon f\|_{t,k,\Omega} \right) + C_t \|N_{q,t}^\epsilon f\|_{t,k-1,\Omega}^2.
\end{aligned}$$

By using a small constant/large constant argument and absorbing terms, we see that

$$(4.15) \quad \sum_{|\alpha| \leq k} Q_{t,\epsilon}(D^\alpha N_{q,t}^\epsilon f, D^\alpha N_{q,t}^\epsilon f) \leq C \|f\|_{t,k,\Omega}^2 + C \|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 + C_t \|N_{q,t}^\epsilon f\|_{t,k-1,\Omega}^2.$$

Proof of Theorem 2.6. The $s = 0$ case for parts (i)-(vii) are the content of the [HRb, Theorem 2.5]. Additionally, as a consequence of the interpolation theory developed for weighted Sobolev spaces on unbounded domains [HR14], it suffices to prove Theorem 2.6 when $s \in \mathbb{N}$. We follow the argument from [HR11, Section 6.4].

Proof of (v.a): Plugging (4.15) into Proposition 3.3, we estimate that for any $\epsilon > 0$

$$(4.16) \quad \sum_{|\alpha| \leq k} \|D^\alpha N_{q,t}^\epsilon f\|_t^2 \leq \epsilon(\|f\|_{t,k,\Omega}^2 + \|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2) + C_t \|N_{q,t}^\epsilon f\|_{t,k-1,\Omega}^2.$$

for all t sufficiently large.

As a consequence of Proposition 4.11, bounding the tangential derivatives suffices to bound the normal derivatives. Thus, (4.16) strengthens to

$$\|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 \leq \epsilon(\|f\|_{t,k,\Omega}^2 + \|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2) + C_t \|N_{q,t}^\epsilon f\|_{t,k-1,\Omega}^2.$$

By choosing ϵ sufficiently small (which forces t to be large), we can absorb terms and establish

$$(4.17) \quad \|N_{q,t}^\epsilon f\|_{t,k,\Omega}^2 \leq C\|f\|_{t,k,\Omega}^2 + C_t \|N_{q,t}^\epsilon f\|_{t,k-1,\Omega}^2$$

where C and C_t are independent of ϵ .

We now let $\epsilon \rightarrow 0$. We have shown that if $f \in W_{0,q}^k(\Omega)$, then $\{N_{q,t}^\epsilon f : 0 < \epsilon < 1\}$ is bounded in $W_{0,q}^k(\Omega)$. This means there exists a sequence $\epsilon_\ell \rightarrow 0$ and $\tilde{u} \in W_{0,q}^k(\Omega)$ so that $N_{q,t}^{\epsilon_\ell} f \rightarrow \tilde{u}$ weakly in $W_{0,q}^k(\Omega)$. Consequently, if $v \in (C_c^\infty)_{0,q}(\Omega)$, then

$$\lim_{\ell \rightarrow \infty} Q_{t,\epsilon_\ell}(N_{q,t}^{\epsilon_\ell} f, v) = Q_t(\tilde{u}, v).$$

Also,

$$Q_{t,\epsilon_\ell}(N_{q,t}^{\epsilon_\ell} f, v) = (f, v) = Q_t(N_{q,t} f, v),$$

so $N_{q,t} f = \tilde{u}$ and (4.17) holds with $\epsilon = 0$. Thus, $N_{q,t}$ is a continuous operator on $W_{0,q}^k(\Omega)$.

Proof of (v.b) and (v.d): The continuity of $\bar{\partial} N_{q,t}$ and $\bar{\partial}_t^* N_{q,t}$ in $W_{0,q}^k(\Omega)$ follows by choosing t larger (possibly). Alternatively, we could modify the argument of [HR11, Section 6.5]. As with $N_{q,t}$, we only need to check that tangential derivatives are bounded.

The remaining items to show are (v.c) and (v.e), the continuity of $N_{q,t} \bar{\partial} : W_{0,q-1}^k \cap \text{Dom}(\bar{\partial}) \rightarrow W_{0,q}^k$ and $N_{q,t} \bar{\partial}_t^* : W_{0,q+1}^k \cap \text{Dom}(\bar{\partial}_t^*) \rightarrow W_{0,q}^k$. Although the case $k = 0$ was done at the end of Section 3 (see also [HR15, Theorem 4.3]), a comment is in order. The operator $N_{q,t} \bar{\partial}_t^*$ appears to require that the form be an element of $\bar{\partial}_t^*$. However, it follows from [HRb, Lemma 3.2] and the fact that $(C_c^\infty)_{0,q+1}(\Omega')$ is dense in $\text{Dom}(\bar{\partial}_t^*)$ on bounded domains Ω' [CS01, Lemma 4.3.2] that $(C_c^\infty)_{0,q+1}(\Omega)$ forms are dense in $\text{Dom}(\bar{\partial}_t^*)$ in $L_{0,q+1}^2(\Omega, \varphi)$. Consequently, the fact that $N_{q,t} \bar{\partial}_t^* = (\bar{\partial} N_{q,t})^*$ is a bounded operator on $L_{0,q+1}^2(\Omega, \varphi)$ means that we can define $N_{q,t} \bar{\partial}_t^*$ on $L_{0,q+1}^2(\Omega, \varphi)$ by density. Thus, $N_{q,t} \bar{\partial}_t^*$ is defined for any form in $L_{0,q+1}^2(\Omega)$, not just forms in $\text{Dom}(\bar{\partial}_t^*)$. A similar argument also justifies writing $N_{q,t} \bar{\partial}$ applied to an arbitrary form in $L_{0,q-1}^2(\Omega)$. Since $N_{q,t} \bar{\partial}_t^*$ and $N_{q,t} \bar{\partial}$ both produce forms in $\text{Dom}(\bar{\partial}_t^*)$, it suffices to estimate the W^k norm using only the special tangential derivatives D^α . To that end, observe that

$$\bar{\partial} D^\alpha N_{q,t} f = [\bar{\partial}, D^\alpha] N_{q,t} f + D^\alpha \bar{\partial} N_{q,t} f$$

and

$$\bar{\partial}_t^* D^\alpha N_{q,t} f = [\bar{\partial}_t^*, D^\alpha] N_{q,t} f + D^\alpha \bar{\partial}_t^* N_{q,t} f.$$

By Proposition 3.3, given $\epsilon > 0$ there exists $t > 0$ so that

$$\|N_{q,t} f\|_{t,k,\Omega}^2 \leq C \sum_{|\alpha| \leq k} \|D^\alpha N_{q,t} f\|_t^2 \leq \epsilon \sum_{|\alpha| \leq k} \left(\|\bar{\partial} D^\alpha N_{q,t} f\|_t^2 + \|\bar{\partial}_t^* D^\alpha N_{q,t} f\|_t^2 \right).$$

Since f has smooth coefficients, by choosing t larger (if necessary), we can use a small constant/large constant argument and estimate

$$\|N_{q,t}f\|_{t,k,\Omega}^2 \leq \epsilon \sum_{|\alpha| \leq k} \left(\|D^\alpha \bar{\partial} N_{q,t}f\|_t^2 + \|D^\alpha \bar{\partial}_t^* N_{q,t}f\|_t^2 \right) + C_t \|N_{q,t}f\|_{t,k-1,\Omega}^2.$$

Next, suppose that $f = \bar{\partial}_t^* g$ for a $(0, q+1)$ -form in $g \in \text{Dom}(\bar{\partial}_t^*)$ with smooth coefficients. Then by induction and [HR11, (22)],

$$\|N_{q,t} \bar{\partial}_t^* g\|_{t,k,\Omega}^2 \leq \epsilon \sum_{|\alpha| \leq k} \|D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_t^2 + C_t \|g\|_{t,k-1,\Omega}^2.$$

We now handle the $\|D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_t^2$ term. Since $D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g \in \text{Dom}(\bar{\partial}_t^*)$, it follows that

$$\begin{aligned} \|D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_t^2 &= (D^\alpha N_{q,t} \bar{\partial}_t^* g, \bar{\partial}_t^* D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g)_t + ([D^\alpha, \bar{\partial}] N_{q,t} \bar{\partial}_t^* g, D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g)_t \\ &\leq |(D^\alpha N_{q,t} \bar{\partial}_t^* g, D^\alpha \bar{\partial}_t^* \bar{\partial} N_{q,t} \bar{\partial}_t^* g)_t| + O\left(\|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega} \|\bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega}\right) \\ &\quad + O_t\left(\|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|-1,\Omega} \|\bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega}\right) \\ &\leq |(D^\alpha N_{q,t} \bar{\partial}_t^* g, D^\alpha \bar{\partial}_t^* g)_t| + l.c. \|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega} + s.c. \|\bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega} + C_t \|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|-1,\Omega}. \end{aligned}$$

Thus,

$$\sum_{|\alpha| \leq k} \|D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_t^2 \leq 2 \sum_{|\alpha| \leq k} |(D^\alpha N_{q,t} \bar{\partial}_t^* g, D^\alpha \bar{\partial}_t^* g)_t| + C \|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega}^2 + C_t \|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|-1,\Omega}^2.$$

Next,

$$\begin{aligned} \sum_{|\alpha| \leq k} (D^\alpha N_{q,t} \bar{\partial}_t^* g, D^\alpha \bar{\partial}_t^* g)_t &= \sum_{|\alpha| \leq k} (D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g, D^\alpha g)_t \\ &\quad + O\left(\|\bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_{t,k,\Omega} \|g\|_{t,k,\Omega}\right) + O_t\left(\|\bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_{t,k,\Omega} \|g\|_{t,k-1,\Omega}\right). \end{aligned}$$

Thus, by absorbing terms after a small constant/large constant argument, we have

$$\sum_{|\alpha| \leq k} \|D^\alpha \bar{\partial} N_{q,t} \bar{\partial}_t^* g\|_t^2 \leq C \|g\|_{t,k,\Omega}^2 + C_t \|g\|_{t,k-1,\Omega}^2 + C \|N_{q,t} \bar{\partial}_t^* g\|_{t,k,\Omega}^2.$$

Finally, by choosing ϵ sufficiently small to absorb the $\|N_{q,t} \bar{\partial}_t^* g\|_{t,|\alpha|,\Omega}^2$ terms, we have proven

$$\|N_{q,t} \bar{\partial}_t^* g\|_{t,k,\Omega}^2 \leq \epsilon \|g\|_{t,k,\Omega}^2 + C_t \|g\|_{t,k-1,\Omega}^2.$$

The argument to prove

$$\|N_{q,t} \bar{\partial} g\|_{t,k,\Omega}^2 \leq \epsilon \|g\|_{t,k,\Omega}^2 + C_t \|g\|_{t,k-1,\Omega}^2$$

is similar.

Proof of (v.g): It suffices to prove the $s = 0$ case for smooth, compactly supported $u \in L_{0,q}^2(\Omega, e^{-t|z|^2})$. Then $\bar{\partial}_t^* \bar{\partial} N_{q,t} u$ is well defined, by (iv). Since $\text{Range } \bar{\partial} \perp \text{Range } \bar{\partial}_t^*$, it follows that

$$\|\bar{\partial}_t^* \bar{\partial} N_{q,t} u\|_t^2 = (\square_{q,t} N_{q,t} u, \bar{\partial}_t^* \bar{\partial} N_{q,t} u)_t = (u, \bar{\partial}_t^* \bar{\partial} N_{q,t} u)_t \leq \|u\|_t \|\bar{\partial}_t^* \bar{\partial} N_{q,t} u\|_t,$$

from which the $s = 0$ case follows. The argument for $\bar{\partial}\bar{\partial}_t^*N_{q,t}u$ is identical. For $s > 0$, we use proof by induction and will show that for any multiindex β of length s ,

$$\|X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u\|_t^2 + \|X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u\|_t^2 \leq C_t\|u\|_{t,s,\Omega}^2.$$

The estimate will follow from the argument of the $s = 0$ case couple with the following estimate.

$$\begin{aligned} & (X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u, X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u)_t \\ &= ([X^\beta, \bar{\partial}_t^*]\bar{\partial}N_{q,t}u, X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u)_t + (\bar{\partial}_t^*X^\beta\bar{\partial}N_{q,t}u, [X^\beta, \bar{\partial}]\bar{\partial}_t^*N_{q,t}u)_t \\ &= ([X^\beta, \bar{\partial}_t^*]\bar{\partial}N_{q,t}u, X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u)_t + ([\bar{\partial}_t^*, X^\beta]\bar{\partial}N_{q,t}u, [X^\beta, \bar{\partial}]\bar{\partial}_t^*N_{q,t}u)_t \\ &\quad + (X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u, [X^\beta, \bar{\partial}]\bar{\partial}_t^*N_{q,t}u)_t \\ &\leq \|\bar{\partial}N_{q,t}u\|_{t,s,\Omega}\|X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u\|_t + \|X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u\|_t\|\bar{\partial}_t^*N_{q,t}u\|_{t,s,\Omega} \\ &\quad + \|\bar{\partial}N_{q,t}u\|_{t,s,\Omega}(\|\bar{\partial}_t^*N_{q,t}u\|_{t,s,\Omega} + C_t\|\bar{\partial}_t^*N_{q,t}u\|_{t,s-1,\Omega}) \end{aligned}$$

Indeed,

$$\begin{aligned} & \|X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u\|_t^2 + \|X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u\|_t^2 = (X^\beta u, X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u)_t + (X^\beta u, X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u)_t \\ & - (X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u, X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u)_t - (X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u, X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u)_t \\ & \leq \|X^\beta u\|_t(\|X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u\|_t + \|X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u\|_t) \\ & + \|\bar{\partial}N_{q,t}u\|_{t,s,\Omega}(\|X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u\|_t + \|\bar{\partial}_t^*N_{q,t}u\|_{t,s,\Omega} + C_t\|\bar{\partial}_t^*N_{q,t}u\|_{t,s-1,\Omega}) \\ & + \|\bar{\partial}_t^*N_{q,t}u\|_{t,s,\Omega}(\|X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u\|_t + \|\bar{\partial}N_{q,t}u\|_{t,s,\Omega}). \end{aligned}$$

From this calculation, it follows that

$$\|X^\beta\bar{\partial}\bar{\partial}_t^*N_{q,t}u\|_t^2 + \|X^\beta\bar{\partial}_t^*\bar{\partial}N_{q,t}u\|_t^2 \leq C\|u\|_{t,s,\Omega}^2 + C_{s,t}\|u\|_{t,s-1,\Omega}^2$$

and consequently half of (v.g) is proven.

Proof of the remaining case of (v.g): By Proposition 4.11, we claim that it suffices to consider derivatives X^β that are tangential near $\text{b}\Omega$. Indeed,

$$\|\bar{\partial}_t^*N_{q,t}\bar{\partial}u\|_{t,k,\Omega}^2 \leq C_k(\|(\nabla_T^t)^k\bar{\partial}_t^*N_{q,t}\bar{\partial}u\|_t^2 + \|\bar{\partial}\bar{\partial}_{q,t}^*N_{q,t}\bar{\partial}u\|_{t,k-1,\Omega}^2) + C_{t,k}\|\bar{\partial}_t^*N_{q,t}\bar{\partial}u\|_{t,k-1,\Omega}^2.$$

However,

$$\|\bar{\partial}\bar{\partial}_{q,t}^*N_{q,t}\bar{\partial}u\|_{t,k-1,\Omega}^2 = \|\bar{\partial}u\|_{t,k-1,\Omega}^2 \leq \|u\|_{t,k,\Omega}^2.$$

Therefore, let T^β be an order k operator that is tangential near $\text{b}\Omega$. The reason that tangential operators are important here is that if $f \in \text{Dom}(\bar{\partial}_t^*)$, then so is $T^\beta f$.

$$\begin{aligned} & \|T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u\|_{t,k,\Omega}^2 \\ &= (T^\beta\bar{\partial}u, T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u)_t + (T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u, [T^\beta, \bar{\partial}_t^*]N_{q,t}\bar{\partial}u)_t + ([\bar{\partial}, T^\beta]\bar{\partial}_t^*N_{q,t}\bar{\partial}u, T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u)_t \\ &= (T^\beta u, T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u)_t + (T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u, [T^\beta, \bar{\partial}_t^*]N_{q,t}\bar{\partial}u)_t + ([\bar{\partial}, T^\beta]\bar{\partial}_t^*N_{q,t}\bar{\partial}u, T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u)_t \\ &\quad + ([T^\beta, \bar{\partial}]u, T^\beta\bar{\partial}_t^*N_{q,t}\bar{\partial}u)_t + (T^\beta u, [\bar{\partial}_t^*, T^\beta]N_{q,t}\bar{\partial}u)_t. \end{aligned}$$

Part (a) now follows from the Cauchy-Schwarz inequality, followed by a small constant/large constant argument and term absorbtion.

Proof if (i)-(iv): This group of results follows from the continuity of the $\bar{\partial}$ -Neumann operator and the canonical solutions operators.

Proof of (vii): We have established the estimates for Kohn's weighted theory, so solvability in C^∞ proceeds using standard arguments. See, for example, [HR11, Section 6.8]. \square

Remark 4.13. Our argument can also be used to establish the following estimate: for $k \geq 1$, there exists $T_k > 0$ so that if $t \geq T_k$, then there exist constants $C_k, C_{k,t} > 0$ where C_s does not depend on t and so that for any $u \in H_{0,q}^k(\Omega, e^{-t|z|^2}, X) \cap \text{Dom}(\bar{\partial}_t^*)$ with $\bar{\partial}u \in H_{0,q+1}^k(\Omega, e^{-t|z|^2}, X)$ and $\bar{\partial}_t^*u \in H_{0,q-1}^k(\Omega, e^{-t|z|^2}, X)$, the inequality

$$\|u\|_{t,k,\Omega}^2 \leq C_k(\|\bar{\partial}u\|_{t,k,\Omega}^2 + \|\bar{\partial}_t^*u\|_{t,k,\Omega}^2) + C_{k,t}\|u\|_{t,k-1,\Omega}^2.$$

holds. The key to show this estimate is Proposition 4.11 which allows us to estimate the norm in $H_{0,q}^k(\Omega, e^{-t|z|^2}, X)$ for forms in $\text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}_t^*)$ with derivatives that are tangential near $\text{b}\Omega$. Consequently, the argument is no different than the $\bar{\partial}_b$ case which we established in [HR11, §6.3]. The result is a matter of careful integration by parts. It is not immediate that this is a closed range estimate for $\bar{\partial}$ (or $\bar{\partial}_t^*$) since $\bar{\partial}_t^*$ is the L^2 -adjoint of $\bar{\partial}$, not the H^k -adjoint.

REFERENCES

- [ABZ06] H. Ahn, L. Baracco, and G. Zampieri. Non-subelliptic estimates for the tangential Cauchy-Riemann system. *Manuscripta Math.*, 121(4):461–479, 2006.
- [AG62] A. Andreotti and H. Grauert. Théorème de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90:193–259, 1962.
- [CS01] S.-C. Chen and M.-C. Shaw. *Partial Differential Equations in Several Complex Variables*, volume 19 of *Studies in Advanced Mathematics*. American Mathematical Society, 2001.
- [FK72] G. B. Folland and J. J. Kohn. *The Neumann problem for the Cauchy-Riemann Complex*, volume 75 of *Ann. of Math. Stud.* Princeton University Press, Princeton, New Jersey, 1972.
- [Gan] K. Gansberger. On the weighted $\bar{\partial}$ -Neumann problem on unbounded domains. arXiv:0912.0841v1.
- [GH10] K. Gansberger and F. Haslinger. Compactness estimates for the $\bar{\partial}$ -Neumann problem in weighted L^2 -spaces. In *Complex analysis*, Trends Math., pages 159–174. Birkhäuser/Springer Basel AG, Basel, 2010. arXiv:0903.1783.
- [HM16] A.-K. Herbig and J. D. McNeal. On closed range for $\bar{\partial}$. *Complex Var. Elliptic Equ.*, 61(8):1073–1089, 2016.
- [Ho91] L.-H. Ho. $\bar{\partial}$ -problem on weakly q -convex domains. *Math. Ann.*, 290(1):3–18, 1991.
- [Hör65] L. Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.
- [HRa] P. Harrington and A. Raich. A remark on boundary estimates on unbounded $Z(q)$ domains in \mathbb{C}^n . *to appear, Complex Var. Elliptic Equ.*
- [HRb] P.S. Harrington and A. Raich. Closed range of $\bar{\partial}$ on unbounded domains in \mathbb{C}^n . *to appear, J. Anal. Math.*
- [HR11] P. Harrington and A. Raich. Regularity results for $\bar{\partial}_b$ on CR-manifolds of hypersurface type. *Comm. Partial Differential Equations*, 36(1):134–161, 2011.
- [HR13] P. Harrington and A. Raich. Defining functions for unbounded C^m domains. *Rev. Mat. Iberoam.*, 29(4):1405–1420, 2013.
- [HR14] P. Harrington and A. Raich. Sobolev spaces and elliptic theory on unbounded domains in \mathbb{R}^n . *Adv. Differential Equations*, 19(7/8):635–692, 2014.
- [HR15] P. Harrington and A. Raich. Closed range for $\bar{\partial}$ and $\bar{\partial}_b$ on bounded hypersurfaces in Stein manifolds. *Ann. Inst. Fourier (Grenoble)*, 65(4):1711–1754, 2015.
- [Koh73] J. J. Kohn. Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds. *Trans. Amer. Math. Soc.*, 181:273–292, 1973.
- [Str10] E. Straube. *Lectures on the \mathcal{L}^2 -Sobolev Theory of the $\bar{\partial}$ -Neumann Problem*. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2010.

[Zam08] G. Zampieri. *Complex analysis and CR geometry*, volume 43 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008.

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